Diploma Programme

Syllabus

Syllabus outline

Syllabus component	Suggested teaching hours—SL	Suggested teaching hours—HL
Topic 1—Number and algebra	16	29
Topic 2—Functions	31	42
Topic 3—Geometry and trigonometry	18	46
Topic 4—Statistics and probability	36	52
Topic 5—Calculus	19	41
The "toolkit" and Mathematical exploration Investigative, problem-solving and modelling skills development leading to an individual exploration. The exploration is a piece of written work that involves investigating an area of mathematics.	30	30
Total teaching hours	150	240

All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as prior learning.

Prior learning topics

Prior to starting a DP mathematics course students have extensive previous mathematical experiences, but these will vary. It is expected that mathematics students will be familiar with the following topics before they take the examinations because questions assume knowledge of them. Teachers must therefore ensure that any topics listed here that are unknown to their students at the start of the course are included at an early stage. Teachers should also take into account the existing mathematical knowledge of their students to design an appropriate course of study for mathematics. This table lists the knowledge, together with the syllabus content, that is essential for successful completion of the mathematics course.

Number and algebra

- Number systems: natural numbers $\mathbb{N}$; integers, $\mathbb{Z}$; rationals, $\mathbb{Q}$, and irrationals; real numbers, $\mathbb{R}$
- SI (Système International) units for mass, time, length, area and volume and their derived units, eg. speed
- Rounding, decimal approximations and significant figures, including appreciation of errors
- Definition and elementary treatment of absolute value (modulus), $|a|$
- Use of addition, subtraction, multiplication and division using integers, decimals and fractions, including order of operations
- Prime numbers, factors (divisors) and multiples
- Greatest common factor (divisor) and least common multiples (HL only)
- Simple applications of ratio, percentage and proportion
- Manipulation of algebraic expressions, including factorization and expansion
- Rearranging formulae
- Calculating the numerical value of expressions by substitution
- Evaluating exponential expressions with simple positive exponents

- Evaluating exponential expressions with rational exponents (HL only)
- Use of inequalities, $<$, $\leq$, $>$, $\geq$, intervals on the real number line
- Simplification of simple expressions involving roots (surds or radicals)
- Rationalising the denominator (HL only)
- Expression of numbers in the form $a \times 10^k$, $1 \leq a < 10$, $k \in \mathbb{Z}$
- Familiarity with commonly accepted world currencies
- Solution of linear equations and inequalities
- Solution of quadratic equations and inequalities with rational coefficients (HL only)
- Solving systems of linear equations in two variables
- Concept and basic notation of sets. Operations on sets: union and intersection
- Addition and subtraction of algebraic fractions (HL only).

Functions

- Graphing linear and quadratic functions using technology
- Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.

Geometry and trigonometry

- Pythagoras' theorem and its converse
- Mid-point of a line segment and the distance between two points in the Cartesian plane
- Geometric concepts: point, line, plane, angle
- Angle measurement in degrees, compass directions
- The triangle sum theorem
- Right-angle trigonometry, including simple applications for solving triangles
- Three-figure bearings
- Simple geometric transformations: translation, reflection, rotation, enlargement
- The circle, its centre and radius, area and circumference. The terms diameter, arc, sector, chord, tangent and segment

- Perimeter and area of plane figures. Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapezoids; compound shapes
- Familiarity with three-dimensional shapes (prisms, pyramids, spheres, cylinders and cones)
- Volumes and surface areas of cuboids, prisms, cylinders, and compound three-dimensional shapes

Statistics and probability

- The collection of data and its representation in bar charts, pie charts, pictograms, and line graphs
- Obtaining simple statistics from discrete data, including mean, median, mode, range
- Calculating probabilities of simple events
- Venn diagrams for sorting data
- Tree diagrams

Calculus

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

Syllabus content

Topic 1: Number and algebra

Concepts

Essential understandings

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

Suggested concepts embedded in this topic

Generalization, representation, modelling, equivalence, approximation, quantity

AHL: Systems, relationships.

Content-specific conceptual understandings:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.

- Different representations of numbers enable quantities to be compared and used for computational purposes with ease and accuracy.
- Numbers and formulae can appear in different, but equivalent forms, or representations, which can help us to establish identities.
- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples
- Mathematical financial models such as compounded growth allow computation, evaluation and interpretation of debt and investment both approximately and accurately.
- Approximation of numbers adds uncertainty or inaccuracy to calculations, leading to potential errors but can be useful when handling extremely large or small quantities.
- Quantities and values can be used to describe key features and behaviours of functions and models, including quadratic functions.

AHL

- Utilizing complex numbers provides a system to efficiently simplify and solve problems.
- Matrices allow us to organize data so that they can be manipulated and relationships can be determined.
- Representing abstract quantities using complex numbers in different forms enables the solution of real-life problems.

SL content

Recommended teaching hours: 16

The aim of the standard level (SL) content of the number and algebra topic is to introduce students to numerical concepts and techniques which combined with an introduction to arithmetic and geometric sequences and series can be used for financial and other applications.

Sections SL1.1 to SL1.5 are content common to Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 1.1

Content	Guidance, clarification and syllabus links
Operations with numbers in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.	Calculator or computer notation is **not** acceptable. For example, 5.2E30 is **not** acceptable and should be written as 5.2×10^{30}.

Connections

Other contexts: Very large and very small numbers, for example astronomical distances, sub-atomic particles in physics, global financial figures

Links to other subjects: Chemistry (Avogadro's number); physics (order of magnitude); biology (microscopic measurements); sciences (uncertainty and precision of measurement)

International-mindedness: The history of number from Sumerians and its development to the present Arabic system

TOK: Do the names that we give things impact how we understand them? For instance, what is the impact of the fact that some large numbers are named, such as the googol and the googolplex, while others are represented in this form?

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SL 1.2

Content	Guidance, clarification and syllabus links
Arithmetic sequences and series. Use of the formulae for the nth term and the sum of the first n terms of the sequence. Use of sigma notation for sums of arithmetic sequences.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. If technology is used in examinations, students will be expected to identify the first term and the common difference.
Applications.	Examples include simple interest over a number of years.
Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life.	Students will need to approximate common differences.

Connections

International-mindedness: The chess legend (Sissa ibn Dahir); Aryabhatta is sometimes considered the “father of algebra”–compare with alKhawarizmi; the use of several alphabets in mathematical notation (for example the use of capital sigma for the sum).

TOK: Is all knowledge concerned with identification and use of patterns? Consider Fibonacci numbers and connections with the golden ratio.

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SL 1.3

Content	Guidance, clarification and syllabus links
Geometric sequences and series. Use of the formulae for the *n*th term and the sum of the first *n* terms of the sequence. Use of sigma notation for the sums of geometric sequences.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. If technology is used in examinations, students will be expected to identify the first term and the ratio. **Link to:** models/functions in topic 2 and regression in topic 4.
Applications.	Examples include the spread of disease, salary increase and decrease and population growth.

Connections

Links to other subjects: Radioactive decay, nuclear physics, charging and discharging capacitors (physics).

TOK: How do mathematicians reconcile the fact that some conclusions seem to conflict with our intuitions? Consider for instance that a finite area can be bounded by an infinite perimeter.

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SL 1.4

Content	Guidance, clarification and syllabus links
Financial applications of geometric sequences and series: • compound interest • annual depreciation.	Examination questions may require the use of technology, including built-in financial packages. The concept of simple interest may be used as an introduction to compound interest. Calculate the real value of an investment with an interest rate and an inflation rate. In examinations, questions that ask students to derive the formula will not be set. Compound interest can be calculated yearly, half-yearly, quarterly or monthly. **Link to**: exponential models/functions in topic 2.

Connections

Other contexts: Loans.

Links to other subjects: Loans and repayments (economics and business management).

Aim 8: Ethical perceptions of borrowing and lending money.

International-mindedness: Do all societies view investment and interest in the same way?

TOK: How have technological advances affected the nature and practice of mathematics? Consider the use of financial packages for instance.

Enrichment: The concept of e can be introduced through continuous compounding, $\left(1+\frac{1}{n}\right)^n \rightarrow \mathrm{e}$, as $n \rightarrow \infty$, however this will not be examined.

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SL 1.5

Content	Guidance, clarification and syllabus links
Laws of exponents with integer exponents.	**Examples:** $5^3 \times 5^{-6} = 5^{-3}$, $6^4 \div 6^3 = 6$, $(2^3)^{-4} = 2^{-12}$, $(2x)^4 = 16x^4$, $2x^{-3} = \frac{2}{x^3}$
Introduction to logarithms with base 10 and e. Numerical evaluation of logarithms using technology.	Awareness that $a^x = b$ is equivalent to $\log_a b = x$, that $b > 0$, and $\log_\mathrm{e} x = \ln x$.

Connections

Other contexts: Richter scale and decibel scale.

Links to other subjects: Calculation of pH and buffer solutions (chemistry)

TOK: Is mathematics invented or discovered? For instance, consider the number e or logarithms–did they already exist before man defined them? (This topic is an opportunity for teachers to generate reflection on “the nature of mathematics”).

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SL 1.6

Content	Guidance, clarification and syllabus links
Approximation: decimal places, significant figures.	Students should be able to choose an appropriate degree of accuracy based on given data.
Upper and lower bounds of rounded numbers.	If $x = 4.1$ to one decimal place, $4.05 \le x < 4.15$.
Percentage errors.	Students should be aware of, and able to calculate, measurement errors (such as rounding errors or measurement limitations). For example finding the maximum percentage error in the area of a circle if the radius measured is 2.5 cm to one decimal place.
Estimation.	Students should be able to recognize whether the results of calculations are reasonable. For example lengths cannot be negative.

Connections

Other contexts: Currency approximations are often to nearest whole number, for example peso, yen; to nearest cent, euro, dollar, pound; meteorology, alternative rounding methods.

Links to other subjects: Order of magnitudes (physics); uncertainty and precision of measurement (sciences).

Aim 8: Caring about approximations; ethical implications.

TOK: Is mathematical reasoning different from scientific reasoning, or reasoning in other areas of knowledge?

Download connections template

SL 1.7

Content	Guidance, clarification and syllabus links
Amortization and annuities using technology.	Technology includes the built-in financial packages of graphic display calculators, spreadsheets. In examinations the payments will be made at the end of the period. Knowledge of the annuity formula will enhance understanding but will not be examined. **Link to**: exponential models (SL 2.5).

Connections

Other contexts: Evaluating the real value of an investment when affected by interest rates and inflation rates. Credit card debt, student loans, retirement planning.

Links to other subjects: Exchange rates (economics), loans (business management).

Aim 8: Ethical perceptions of borrowing and lending money; short-term loans at high interest rates, how can knowledge of mathematics result in individuals being exploited or protected from extortion?

International-mindedness: Do all societies view investment and interest in the same way?

Download connections template

SL 1.8

Content	Guidance, clarification and syllabus links
Use technology to solve: • Systems of linear equations in up to 3 variables • Polynomial equations	In examinations, no specific method of solution will be required. In examinations, there will always be a unique solution to a system of equations. Standard terminology, such as zeros or roots, should be taught. **Link to:** quadratic models (SL 2.5)

Connections

Links to other subjects: Kirchhoff's laws (physics).

TOK: What role does language play in the accumulation and sharing of knowledge in mathematics? Consider for example that when mathematicians talk about "imaginary" or "real" solutions they are using precise technical terms that do not have the same meaning as the everyday terms.

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AHL content

Recommended teaching hours: 13

The aim of the AHL content in the number and algebra topic is to extend and build upon the aims, concepts and skills from the standard level content. It introduces students to the laws of logarithms, the important mathematical concepts of complex numbers and matrices, and their applications.

AHL 1.9

Content	Guidance, clarification and syllabus links
Laws of logarithms: $\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m\log_a x$ for $a, \quad x, \quad y \quad > \quad 0$	In examinations, a will equal 10 or e. **Link to:** scaling large and small numbers (AHL 2.10).

Connections

Links to other subjects: pH, buffer calculations and finding activation energy from experimental data (chemistry).

TOK: What is meant by the terms “law” and “theory” in mathematics. How does this compare to how these terms are used in different areas of knowledge?

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AHL 1.10

Content	Guidance, clarification and syllabus links
Simplifying expressions, both numerically and algebraically, involving rational exponents.	**Examples:** $5^{\frac{1}{2}} \times 5^{\frac{1}{3}} = 5^{\frac{5}{6}}, \ 6^{\frac{3}{4}} \div 6^{\frac{1}{2}} = 6^{\frac{1}{4}}, \ 32^{\frac{3}{5}} = 8, \ x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

Connections

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AHL 1.11

Content	Guidance, clarification and syllabus links
The sum of infinite geometric sequences.	**Link to:** the concept of a limit (SL 5.1), fractals (AHL 3.9), and Markov chains (AHL 4.19).

Connections

Other contexts: Total distance travelled by a bouncing ball.

TOK: Is it possible to know about things of which we can have no experience, such as infinity?

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AHL 1.12

Content	Guidance, clarification and syllabus links
Complex numbers: the number i such that $\mathrm{i}^2 = -1$. Cartesian form: $z = a + b\mathrm{i}$; the terms real part, imaginary part, conjugate, modulus and argument. Calculate sums, differences, products, quotients, by hand and with technology. Calculating powers of complex numbers, in Cartesian form, with technology.	
The complex plane. Complex numbers as solutions to quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$, with real coefficients where $b^2 - 4ac < 0$.	Use and draw Argand diagrams. Quadratic formula and the link with the graph of $f(x) = ax^2 + bx + c$.

Connections

TOK: How does language shape knowledge? For example do the words “imaginary” and “complex” make the concepts more difficult than if they had different names?

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AHL 1.13

Content	Guidance, clarification and syllabus links
Modulus–argument (polar) form: $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$.	
Exponential form: $z = r\mathrm{e}^{\mathrm{i}\theta}$.	Exponential form is sometimes called the Euler form.
Conversion between Cartesian, polar and exponential forms, by hand and with technology.	
Calculate products, quotients and integer powers in polar or exponential forms.	In examinations students will not be required to find the roots of complex numbers.
Adding sinusoidal functions with the same frequencies but different phase shift angles.	Phase shift and voltage in circuits as complex quantities. **Example:** Two AC voltages sources are connected in a circuit. If $V_1 = 10\cos(40t)$ and $V_2 = 20\cos(40t + 10)$ find an expression for the total voltage in the form $V = A\cos(40t + B)$.
Geometric interpretation of complex numbers.	Addition and subtraction of complex numbers can be represented as vector addition and subtraction. Multiplication of complex numbers can be represented as a rotation and a stretch in the Argand diagram.

Connections

TOK: Why might it be said that $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$ is beautiful? What is the place of beauty and elegance in mathematics? What about the place of creativity?

Enrichment: Solution of differential equations by separation of variables (AHL5.15), as both polar and exponential forms are solutions of $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$.

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AHL 1.14

Content	Guidance, clarification and syllabus links
Definition of a matrix: the terms element, row, column and order for $m \times n$ matrices.	
Algebra of matrices: equality; addition; subtraction; multiplication by a scalar for $m \times n$ matrices.	Including use of technology.
Multiplication of matrices. Properties of matrix multiplication: associativity, distributivity and non-commutativity.	Multiplying matrices to solve practical problems.
Identity and zero matrices. Determinants and inverses of $n \times n$ matrices with technology, and by hand for 2×2 matrices.	Students should be familiar with the notation ***I*** and ***0***.
Awareness that a system of linear equations can be written in the form $\boldsymbol{A}x = b$.	In examinations ***A*** will always be an invertible matrix, except when solving for eigenvectors.
Solution of the systems of equations using inverse matrix.	Model and solve real-life problems including: Coding and decoding messages Solving systems of equations. **Link to:** Markov chains (AHL 4.19), transition matrices (AHL 4.19) and phase portrait (AHL 5.17).

Connections

Other contexts: Comparing sales/revenue/profit for multiple products over multiple weeks.

TOK: Given the many applications of matrices in this course, consider the fact that mathematicians marvel at some of the deep connections between disparate parts of their subject. Is this evidence for a simple underlying mathematical reality? Mathematics, sense, perception and reason–if we can find solutions of higher dimensions, can we reason that these spaces exist beyond our sense perception?

External link: Simulation for encoding and decoding messages using various methods including exercises:

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AHL 1.15

Content	**Guidance, clarification and syllabus links**
Eigenvalues and eigenvectors. Characteristic polynomial of 2×2 matrices. Diagonalization of 2×2 matrices (restricted to the case where there are distinct real eigenvalues).	Students will only be expected to perform calculations by hand and with technology for 2×2 matrices.
Applications to powers of 2×2 matrices.	Applications, for example movement of population between two towns, predator/prey models. $\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors, and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues. **Link to:** coupled differential equations (AHL 5.17).

Connections

Other contexts: Invariant states; representation of conics.

Links to other subjects: Stochastic processes, stock market values and trends (business management).

Aim 8: Damping noise in car design, test for cracks in solid objects, oil exploration, the Google PageRank formula, and “the $25 billion dollar eigenvector”; the natural frequency of an object can be characterized by the eigenvalue of smallest magnitude (1940 Tacoma Narrows bridge disaster).

TOK: Mathematics can be used successfully to model real-world processes. Is this because mathematics was created to mirror the world or because the world is intrinsically mathematical?

Enrichment: Principal component and factor analysis. Link between discrete change and continuous change in dynamical systems (including why e is such an important number).

Download connections template

Topic 2: Functions

Concepts

Essential understandings:

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variable. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

Suggested concepts embedded in this topic:

Representation, relationships, space, modelling, change.

AHL: Generalization, validity.

Content-specific conceptual understandings:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.

- The parameters in a function or equation may correspond to notable geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Technology plays a key role in allowing humans to represent the real world as a model and to quantify the appropriateness of the model.

AHL

- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.
- Generalization provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.
- Considering the reasonableness and validity of results helps us to make informed, unbiased decisions.

SL content

Recommended teaching hours: 31

The aim of the standard level content in the functions topic is to introduce students to the important unifying theme of a function in mathematics and the skills needed to model and interpret practical situations with a variety of key functions.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of functions, rather than using elaborate analytical techniques.

On examination papers:

- questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus
- the domain will be the largest possible domain for which a function is defined unless otherwise stated; this will usually be the real numbers.

Sections SL2.1 to SL2.4 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 2.1

Content	Guidance, clarification and syllabus links
Different forms of the equation of a straight line. Gradient; intercepts. Lines with gradients m_1 and m_2 Parallel lines $m_1 = m_2$. Perpendicular lines $m_1 \times m_2 = -1$.	$y = mx + c$ (gradient-intercept form). $ax + by + d = 0$ (general form). $y - y_1 = m(x - x_1)$ (point-gradient form). Calculate gradients of inclines such as mountain roads, bridges, etc.

Connections

Other contexts: Gradients of mountain roads, gradients of access ramps.

Links to other subjects: Exchange rates and price and income elasticity, demand and supply curves (economics); graphical analysis in experimental work (sciences).

TOK: Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

Download connections template

SL 2.2

Content	Guidance, clarification and syllabus links
Concept of a function, domain, range and graph. Function notation, for example $f(x),\ v(t),\ C(n)$. The concept of a function as a mathematical model.	**Example:** $f(x)=\sqrt{2-x}$, the domain is $x\leq 2$, range is $f(x)\geq 0$. A graph is helpful in visualizing the range.
Informal concept that an inverse function reverses or undoes the effect of a function. Inverse function as a reflection in the line $y=x$, and the notation $f^{-1}(x)$.	**Example:** Solving $f(x)=10$ is equivalent to finding $f^{-1}(10)$. Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$ is equal to the range of $f(x)$.

Connections

Other contexts: Temperature and currency conversions**.**

Links to other subjects: Currency conversions and cost functions (economics and business management); projectile motion (physics).

Aim 8: What is the relationship between real-world problems and mathematical models?

International-mindedness: The development of functions by Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland); the notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries–how did the notation we use today become internationally accepted?

TOK: Do you think mathematics or logic should be classified as a language?

Download connections template

SL 2.3

Content	Guidance, clarification and syllabus links
The graph of a function; its equation $y = f(x)$.	Students should be aware of the difference between the command terms "draw" and "sketch".
Creating a sketch from information given or a context, including transferring a graph from screen to paper. Using technology to graph functions including their sums and differences.	All axes and key features should be labelled. This may include functions not specifically mentioned in topic 2.

Connections

Links to other subjects: Sketching and interpreting graphs (sciences, geography, economics).

TOK: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

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SL 2.4

Content	Guidance, clarification and syllabus links
Determine key features of graphs.	Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology.
Finding the point of intersection of two curves or lines using technology.	

Connections

Links to other subjects: Identification and interpretation of key features of graphs (sciences, geography, economics); production possibilities curve model, market equilibrium (economics).

International-mindedness: Bourbaki group analytical approach versus the Mandlebrot visual approach.

Use of technology: Graphing technology with sliders to determine the effects of altering parameters and variables.

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SL 2.5

Content	Guidance, clarification and syllabus links
Modelling with the following functions:	
Linear models. $f(x) = mx + c$.	Including piecewise linear models, for example horizontal distances of an object to a wall, depth of a swimming pool, mobile phone charges. **Link to:** equation of a straight line (SL 2.1) and arithmetic sequences (SL 1.2).
Quadratic models. $f(x) = ax^2 + bx + c \;\;;\;\; a \neq 0$. Axis of symmetry, vertex, zeros and roots, intercepts on the x-axis and y -axis.	Technology can be used to find roots. **Link to:** use of technology to solve quadratic equations (SL 1.8).
Exponential growth and decay models. $f(x) = ka^x + c$ $f(x) = ka^{-x} + c \quad (\text{for } a > 0)$ $f(x) = ke^{rx} + c$ Equation of a horizontal asymptote.	**Link to:** compound interest (SL 1.4), geometric sequences and series (SL 1.3) and amortization (SL 1.7).
Direct/inverse variation: $f(x) = ax^n, \quad n \in \mathbb{Z}$ The y-axis as a vertical asymptote when $n < 0$.	
Cubic models:	

$f(x)=ax^3+bx^2+cx+d$.	
Sinusoidal models: $f(x)=a\sin(bx)+d, \quad f(x)=a\cos(bx)+d$.	Students will **not** be expected to translate between $\sin x$ and $\cos x$, and will only be required to predict or find amplitude (a), period $\left(\frac{360^\circ}{b}\right)$, or equation of the principal axis ($y=d$).

Connections

Other contexts:

Linear models: Conversion graphs, for example Fahrenheit to Celsius, currency. Cost of hiring an item at a daily rate with a fixed deposit.

Quadratic models: Cost functions, satellite dishes, bridges, projectile motion.

Exponential models: Population growth, radioactive decay, cooling of a liquid, spread of a virus, compound interest, depreciation and amortization.

Direct/inverse variation: Boyle's law and Charles's law of gases, laws of supply and demand.

Cubic models: The volume of a box with a fixed surface area, the amount of wasted space in a can of tennis balls, power produced by a wind turbine and wind speed.

Sinusoidal models: Periodic phenomena that give rise to sinusoidal models, for example tides, weather patterns, motion of ferris and bicycle wheels, annual temperatures.

Links to other subjects: Population growth, spread of a virus (biology); radioactive decay and half-life, X-ray attenuation, cooling of a liquid, kinematics, simple harmonic motion, projectile motion, inverse square law (physics); compound interest, depreciation (business management); the circular flow of income model (economics); the equilibrium law and rates of reaction (chemistry); opportunities to model as part of experimental work (science)

Aim 8: The phrase "exponential growth" is used popularly to describe a number of phenomena. Is this a misleading use of the mathematical term?

International-mindedness: The Babylonian method of multiplication: $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

TOK: What role do models play in mathematics? Do they play a different role in mathematics compared to their role in other areas of knowledge?

Use of technology: Generating parabolas using dynamic geometry software.

Enrichment: Conics–how can a parabola be created by cutting a cone?

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SL 2.6

Content	Guidance, clarification and syllabus links
Modelling skills: Use the modelling process described in the "mathematical modelling" section to create, fit and use the theoretical models in section SL2.5 and their graphs.	Fitting models using regression is covered in topic 4. **Link to**: theoretical models (SL 2.5) to be used to develop the modelling skills and, for HL students, (AHL 2.9).
Develop and fit the model: Given a context recognize and choose an appropriate model and possible parameters. Determine a reasonable domain for a model.	
Find the parameters of a model.	By setting up and solving equations simultaneously (using technology), by consideration of initial conditions or by substitution of points into a given function. At SL, students will not be expected to perform non-linear regressions, but will be expected to set up and solve up to three linear equations in three variables using technology.
Test and reflect upon the model: Comment on the appropriateness and reasonableness of a model.	

Justify the choice of a particular model, based on the shape of the data, properties of the curve and/or on the context of the situation.	
Use the model: Reading, interpreting and making predictions based on the model.	Students should be aware of the dangers of extrapolation.

Connections

Links to other subjects: opportunities to model as part of experimental work (science).

TOK: What is it about models in mathematics that makes them effective? Is simplicity a desirable characteristic in models?

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AHL content

Recommended teaching hours: 11

The aim of the AHL functions topic is to extend the aims, concepts and skills from the standard level content. It introduces students to further numerical and graphical techniques and further key functions which can be used to model and interpret practical situations.

AHL 2.7

Content	**Guidance, clarification and syllabus links**
Composite functions in context. The notation $(f \circ g)(x) = f(g(x))$. Inverse function f^{-1}, including domain restriction. Finding an inverse function.	$\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x.$ **Example:** $f(x) = (x-3)^2 - 2$ has an inverse if the domain is restricted to $x \geq 3$ or to $x \leq 3$.

Connections

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AHL 2.8

Content	Guidance, clarification and syllabus links
Transformations of graphs.	Students will be expected to be able to perform transformations on all functions from the SL and AHL section of this topic, and others in the context of modelling real-life situations.
Translations: $y=f(x)+b; y=f(x-a)$. Reflections: in the x axis $y=-f(x)$, and in the y axis $y=f(-x)$.	Translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ denotes horizontal translation of 3 units to the right, and vertical translation of 2 units down.
Vertical stretch with scale factor p: $y=pf(x)$. Horizontal stretch with scale factor $\frac{1}{q}$: $y=f(qx)$	x and y axes are invariant.
Composite transformations.	Students should be made aware of the significance of the order of transformations. **Example:** $y=x^2$ used to obtain $y=3x^2+2$ by a vertical stretch of scale factor 3 followed by a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. **Example:** $y=\sin x$ used to obtain $y=4\sin 2x$ by a vertical stretch of scale factor 4 and a horizontal stretch of scale factor $\frac{1}{2}$.

Connections

Other contexts: Translating curves to reduce rounding errors for large values.

Links to other subjects: Shifting of supply and demand curves (economics); electromagnetic induction (physics)

TOK: Is mathematics independent of culture? To what extent are we aware of the impact of culture on what we believe or know?

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AHL 2.9

Content	Guidance, clarification and syllabus links
In addition to the models covered in the SL content the AHL content extends this to include modelling with the following functions: Exponential models to calculate half-life.	**Link to:** modelling skills (SL2.6).
Natural logarithmic models: $f(x) = a + b\ln x$	
Sinusoidal models: $f(x) = a\sin(b(x-c)) + d$	Radian measure should be assumed unless otherwise indicated by the use of the degree symbol, for example with $f(x) = \sin x°$. In radians, period is $\frac{2\pi}{b}$. Students should be aware that a horizontal translation of c can be referred to as a phase shift. **Link to:** radian measure (AHL 3.7)
Logistic models: $f(x) = \frac{L}{1+Ce^{-kx}}; L, \quad C, \quad k > 0$	The logistic function is used in situations where there is a restriction on the growth. For example population on an island, bacteria in a petri dish or the increase in height of a person or seedling. Horizontal asymptote at $f(x) = L$ is often referred to as the carrying capacity.
Piecewise models.	In some cases, parameters may need to be found that ensure continuity of the function, for

	example find a to make $f(x)=\begin{cases}1+x, & 0\le x<2\\ ax^2+x, & x\ge 2\end{cases}$ continuous. The formal definition of continuity is not required. In examinations, students may be expected to interpret and use other models that are introduced in the question.

Connections

Other contexts:

Sinusoidal models: Waxing and waning of the moon, rainfall patterns, temperature, movement of bridges and buildings, pH scale, Richter scale, sound intensity, brightness of stars.

Piecewise models: Income taxes and taxi fares, friction, mobile phone plans, depth of pool as function of distance from the deep end (piecewise linear), postage rates for letters, stock prices, parachuting before and after the chute opens, Hooke's law, shapes of buildings, horizontal distance from a wall to a curved object.

Links to other subjects: Half-life (chemistry and physics); AC circuits and waves (physics); the Gini coefficient and the Lorenz curve, and progressive, regressive and proportional taxes, the J-curve (economics)

TOK: Is there a hierarchy of areas of knowledge in terms of their usefulness in solving problems?

Enrichment: For the population equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{L}\right)$ with $P=P_0$ when $t=0$, the solution is the logistic equation: $P=\frac{L}{1+C\mathrm{e}^{-kt}}$, with $C=\frac{L}{P_0}-1$.

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AHL 2.10

Content	Guidance, clarification and syllabus links
Scaling very large or small numbers using logarithms. Linearizing data using logarithms to determine if the data has an exponential or a power relationship using best-fit straight lines to determine parameters.	Choosing a manageable scale, for example for data with a wide range of values in one, or both variables and/or where the emphasis of a graph is the rate of growth, rather than the absolute value. **Link to:** laws of logarithms (AHL 1.9) and Pearson's product moment correlation coefficient (SL 4.4).
Interpretation of log-log and semi-log graphs.	In examinations, students will not be expected to draw or sketch these graphs.

Connections

Other contexts: Growth of bacteria or traffic to websites/social media; exponential graphs that show alarming absolute figures, but reasonable rates of growth.

Links to other subjects: pH semi-log curves and finding activation energy from experimental data (chemistry); exponential decay (physics); experimental work (sciences).

TOK: Does the applicability of knowledge vary across the different areas of knowledge? What would the implications be if the value of all knowledge was measured solely in terms of its applicability?

Links to websites: Gapminder makes use of log-log graphs: www.gapminder.org

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Topic 3: Geometry and trigonometry

Concepts

Essential understandings

Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Suggested concepts embedded in this topic

Generalization, space, relationships, systems, representations

AHL: Quantity, change.

Content-specific conceptual understandings:

- The properties of shapes are highly dependent on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Different representations of trigonometric expressions help to simplify calculations.
- Systems of equations often, but not always, lead to intersection points.
- In two dimensions, the Voronoi diagram allows us to navigate, path-find or establish an optimum position.

AHL

- Different measurement systems can be used for angles to facilitate ease of calculation.
- Vectors allow us to determine position, change of position (movement) and force in two and three-dimensional space.
- Graph theory algorithms allow us to represent networks and to model complex real-world problems.
- Matrices are a form of notation which allow us to show the parameters or quantities of several linear equations simultaneously.

SL content

Recommended teaching hours: 18

The aim of the standard level content of the geometry and trigonometry topic is to introduce students to appropriate skills and techniques for practical problem solving in two and three dimensions.

Throughout this topic students should be given the opportunity to use technology such as graphing packages, graphing calculators and dynamic geometry software to develop and apply their knowledge of geometry and trigonometry.

Sections SL3.1 to SL3.3 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 3.1

Content	Guidance, clarification and syllabus links
The distance between two points in three-dimensional space, and their midpoint. Volume and surface area of three-dimensional solids including right-pyramid, right cone, sphere, hemisphere and combinations of these solids. The size of an angle between two intersecting lines or between a line and a plane.	In SL examinations, only right-angled trigonometry questions will be set in reference to three-dimensional shapes. In problems related to these topics, students should be able to identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles.

Connections

Other contexts: Architecture and design.

Links to other subjects: Design technology; volumes of stars and inverse square law (physics).

TOK: What is an axiomatic system? Are axioms self evident to everybody?

Download connections template

SL 3.2

Content	Guidance, clarification and syllabus links
Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles.	In all areas of this topic, students should be encouraged to sketch well-labelled diagrams to support their solutions. **Link to:** inverse functions (SL2.2) when finding angles.
The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ Area of a triangle as $\frac{1}{2}ab\sin C$.	This section does not include the ambiguous case of the sine rule.

Connections

Other contexts: Triangulation, map-making.

Links to other subjects: Vectors (physics).

International-mindedness: Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics; the use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation? What criteria might we use to make such a judgment?

Download connections template

SL 3.3

Content	Guidance, clarification and syllabus links
Applications of right and non-right angled trigonometry, including Pythagoras' theorem. Angles of elevation and depression. Construction of labelled diagrams from written statements.	Contexts may include use of bearings.

Connections

Other contexts: Triangulation, map-making, navigation and radio transmissions. Use of parallax for navigation.

Links to other subjects: Vectors, scalars, forces and dynamics (physics); field studies (sciences)

Aim 8: Who really invented Pythagoras' theorem?

Aim 9: In how many ways can you prove Pythagoras' theorem?

International-mindedness: The use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: If the angles of a triangle can add up to less than 180°, 180° or more than 180°, what does this tell us about the nature of mathematical knowledge?

Download connections template

SL 3.4

Content	Guidance, clarification and syllabus links
The circle: length of an arc; area of a sector.	Radians are not required at SL.

Connections

TOK: Does personal experience play a role in the formation of knowledge claims in mathematics? Does it play a different role in mathematics compared to other areas of knowledge?

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SL 3.5

Content	Guidance, clarification and syllabus links
Equations of perpendicular bisectors.	Given either two points, or the equation of a line segment and its midpoint. **Link to:** equations of straight lines (SL 2.1).

Connections

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SL 3.6

Content	Guidance, clarification and syllabus links
Voronoi diagrams: sites, vertices, edges, cells. Addition of a site to an existing Voronoi diagram. Nearest neighbour interpolation. Applications of the "toxic waste dump" problem.	In examinations, coordinates of sites for calculating the perpendicular bisector equations will be given. Students will not be required to construct perpendicular bisectors. Questions may include finding the equation of a boundary, identifying the site closest to a given point, or calculating the area of a region. All points within a cell can be estimated to have the same value (e.g. rainfall) as the value of the site. In examinations, the solution point will always be at an intersection of three edges. **Contexts:** Urban planning, spread of diseases, ecology, meteorology, resource management.

Connections

Other contexts: Applications in subjects including geography, economics, biology, and computer science. www.ics.uci.edu/~eppstein/gina/scot.drysdale.html

TOK: Is the division of knowledge into disciplines or areas of knowledge artificial?

Link to TSM: Incremental algorithm for constructing Voronoi diagrams.

Enrichment: Delaunay triangulations as the duals of Voronoi triangulations; self-driving cars; the art gallery problem. Natural neighbour interpolation. Manhattan metric.

Download connections template

AHL Content

Recommended teaching hours: 28

The aim of the AHL content in the geometry and trigonometry topic is to extend and build upon the aims, concepts and skills from the standard level content. It introduces students to an alternative measurement system for angles and some important trigonometric identities, extends the application of matrices to transformations, and introduces students to vectors and their applications in kinematics. Graph theory is introduced to allow students to apply their knowledge of matrices and develop their knowledge of algorithms in practical contexts.

On HL examination papers radian measure should be assumed unless otherwise indicated.

AHL 3.7

Content	Guidance, clarification and syllabus links
The definition of a radian and conversion between degrees and radians. Using radians to calculate area of sector, length of arc.	Radian measure may be expressed as exact multiples of π, or decimals. **Link to**: trigonometric functions (AHL 2.9).

Connections

Links to other subjects: Diffraction patterns and circular motion (physics)

International-mindedness: Seki Takakazu calculating π to ten decimal places; Hipparchus, Menelaus and Ptolemy; why are there 360 degrees in a complete turn? Why do we use minutes and seconds for time?; Links to Babylonian mathematics.

TOK: Which is the better measure of an angle, degrees or radians? What criteria can/do/should mathematicians use to make such judgments?

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AHL 3.8

Content	Guidance, clarification and syllabus links
The definitions of $\cos\theta$ and $\sin\theta$ in terms of the unit circle. The Pythagorean identity: $\cos^2\theta+\sin^2\theta=1$ Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$ Extension of the sine rule to the ambiguous case.	Students should understand how the graphs of $f(x)=\sin x$ and $f(x)=\cos x$ can be constructed from the unit circle. Knowledge of exact values of $\cos\theta$, $\sin\theta$, and $\tan\theta$ will not be assessed on examinations, but may aid student understanding of trigonometric functions.
Graphical methods of solving trigonometric equations in a finite interval.	**Link to:** sinusoidal models (SL2.5 and AHL2.9).

Connections

Other contexts: Generation of sinusoidal voltage in electrical engineering.

International-mindedness: The origin of the word “sine”; trigonometry was developed by successive civilizations and cultures; how is mathematical knowledge considered from a sociocultural perspective?

TOK: To what extent is mathematical knowledge embedded in particular traditions or bound to particular cultures? How have key events in the history of mathematics shaped its current form and methods?

Use of technology: Animation applets that show the development of a trigometric function graph from the unit circle.

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AHL 3.9

Content	Guidance, clarification and syllabus links
Geometric transformations of points in two dimensions using matrices: reflections, horizontal and vertical stretches, enlargements, translations and rotations.	Matrix transformations of the form: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix}$. **Link to:** matrices (AHL 1.14).
Compositions of the above transformations.	Iterative techniques to generate fractals. **Link to:** infinite geometric series (AHL 1.11) and Markov chains (AHL 4.19).
Geometric interpretation of the determinant of a transformation matrix.	Area of image $=\lvert \det \boldsymbol{A} \rvert \times$ area of object.

Connections

Other contexts: Fractals: “mutations” in biology–changing the probability with which different “matrix transformations” occur [or changing the initial value ($Z_0 = c$) in the Mandelbrot quadratic recurrence equation: $Z_{n+1} = {Z_n}^2 + c$] and then observing what part of the structure/form this affects.

Sierpinski’s triangle and Koch’s snowflake make a good introduction to algorithms that generate fractals.

Aim 8: Matrices used in computer graphics for three-dimensional modelling: how has this been used to advance diagnoses of health conditions?

TOK: When mathematicians and historians say that they have explained something, are they using the word “explain” in the same way?

Website: Useful examples of fractal applications: http://www.fractal.org/Bewustzijns-Besturings-Model/Fractals-Useful-Beauty.htm

Enrichment: Affine transformations and digital image processing.

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AHL 3.10

Content	Guidance, clarification and syllabus links
Concept of a vector and a scalar. Representation of vectors using directed line segments. Unit vectors; base vectors $\boldsymbol{i}, \quad \boldsymbol{j}, \quad \boldsymbol{k}$. Components of a vector; column representation; $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ The zero vector $\boldsymbol{0}$, the vector $-\boldsymbol{v}$. Position vectors $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$.	Use algebraic and geometric approaches to calculate the sum and difference of two vectors, multiplication by a scalar, $k\boldsymbol{v}$ (parallel vectors), magnitude of a vector $\lvert\boldsymbol{v}\rvert$ from components. The resultant as the sum of two or more vectors.
Rescaling and normalizing vectors.	$\frac{\boldsymbol{v}}{\lvert\boldsymbol{v}\rvert}$, the unit normal vector. **Example:** Find the velocity of a particle with speed 7ms$_{-1}$ in the direction $3\boldsymbol{i} + 4\boldsymbol{j}$.

Connections

Links to other subjects: Vector sums, differences and resultants (physics).

Aims: Vector theory is used for tracking displacement of objects, including peaceful and harmful purposes.

TOK: Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb. To what extent does possession of knowledge carry with it an ethical obligation?

Download connections template

AHL 3.11

Content	**Guidance, clarification and syllabus links**
Vector equation of a line in two and three dimensions: $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$, where $\boldsymbol{b}$ is a direction vector of the line.	Convert to parametric form: $x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n.$

Connections

TOK: Mathematics and the knower: Why are symbolic representations of three-dimensional objects easier to deal with than visual representations? What does this tell us about our knowledge of mathematics in other dimensions?

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AHL 3.12

Content	Guidance, clarification and syllabus links
Vector applications to kinematics. Modelling linear motion with constant velocity in two and three dimensions.	Finding positions, intersections, describing paths, finding times and distances when two objects are closest to each other. $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}t$. Relative position of B from A is $\overrightarrow{AB}$.
Motion with variable velocity in two dimensions.	**For example:** $\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 7 \\ 6 - 4t \end{pmatrix}$. Projectile motion and circular motion are special cases. $f(t-a)$ to indicate a time-shift of a. **Link to:** kinematics (AHL 5.13) and phase shift (AHL 1.13).

Connections

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AHL 3.13

Content	Guidance, clarification and syllabus links
Definition and calculation of the scalar product of two vectors. The angle between two vectors; the acute angle between two lines.	Calculate the angle between two vectors using $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between two non-zero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, and ascertain whether the vectors are perpendicular ($\boldsymbol{v} \cdot \boldsymbol{w} = 0$).
Definition and calculation of the vector product of two vectors.	$\boldsymbol{v} \times \boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta\boldsymbol{n}$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ and $\boldsymbol{n}$ is the unit normal vector whose direction is given by the right-hand screw rule. **Not required:** generalized properties and proofs of scalar and cross product.
Geometric interpretation of $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert$.	Use of $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert$ to find the area of a parallelogram (and hence a triangle).
Components of vectors.	The component of vector $\boldsymbol{a}$ acting in the direction of vector $\boldsymbol{b}$ is $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{b}\rvert} = \lvert\boldsymbol{a}\rvert\cos\theta$. The component of a vector $\boldsymbol{a}$ acting perpendicular to vector $\boldsymbol{b}$, in the plane formed by the two vectors, is $\frac{\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert}{\lvert\boldsymbol{b}\rvert} = \lvert\boldsymbol{a}\rvert\sin\theta$.

Connections

Other contexts: *Computer graphics*: Lighting–normalize one vector onto another to determine the intensity of light on a surface. Perspective–project a three-dimensional vector onto a two-dimensional plane using the scalar product.

Physics: Torque– magnitude of the rotational force ('torque') applied to a point/object is equal to the magnitude of the cross product of the "length" of the lever and the "force" applied to that level. The direction of the torque, for example will the force twist the the nut "on" (tightening) or "off" (loosening) the bolt, is the cross product. Electro-magnetic forces and the right and left hand rules. $W = F \cdot d$. Forces–what component of a vector's force is acting in the direction of another–useful for any "structural" analysis when working out the "strain" on each part of the structure that results from a given force.

Links to other subjects: Magnetic forces and fields, and dynamics (physics)

TOK: What counts as understanding in mathematics? Is it more than just getting the right answer?

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AHL 3.14

Content	Guidance, clarification and syllabus links
Graph theory: Graphs, vertices, edges, adjacent vertices, adjacent edges. Degree of a vertex.	Students should be able to represent real-world structures (circuits, maps, etc) as graphs (weighted and unweighted).
Simple graphs; complete graphs; weighted graphs.	Knowledge of the terms connected and strongly connected.
Directed graphs; in degree and out degree of a directed graph. Subgraphs; trees.	**Link to**: matrices (AHL 1.14).

Connections

Aim 8: The importance of symbolic maps, for example Metro and Underground maps, structural formulae in chemistry, electrical circuits.

TOK: Mathematics and knowledge claims. Proof of the four-colour theorem. If a theorem is proved by computer, how can we claim to know that it is true?

Download connections template

AHL 3.15

Content	Guidance, clarification and syllabus links
Adjacency matrices. Walks. Number of k -length walks (or less than k -length walks) between two vertices.	Given an adjacency matrix $\boldsymbol{A}$, the $(i,\quad j)_{\text{th}}$ entry of $\boldsymbol{A}^k$ gives the number of k length walks connecting i and j.
Weighted adjacency tables. Construction of the transition matrix for a strongly-connected, undirected or directed graph.	Weights could be costs, distances, lengths of time for example. Consideration of simple graphs, including the Google PageRank algorithm as an example of this. **Link to:** transition matrices and Markov chains (AHL 4.19).

Connections

International-mindedness: The “Bridges of Konigsberg” problem.

Links to websites: Adjacency matrix and airlines:

Page rank is one method used to determine the importance rank of a webpage. Simulation for PageRank: www.eprisner.de/MAT103/PageRank.html

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AHL 3.16

Content	Guidance, clarification and syllabus links
Tree and cycle algorithms with undirected graphs. Walks, trails, paths, circuits, cycles.	
Eulerian trails and circuits. Hamiltonian paths and cycles. Minimum spanning tree (MST) graph algorithms: Kruskal's and Prim's algorithms for finding minimum spanning trees.	Determine whether an Eulerian trail or circuit exists. Use of matrix method for Prim's algorithm.
Chinese postman problem and algorithm for solution, to determine the shortest route around a weighted graph with up to four odd vertices, going along each edge at least once.	Students should be able to explain why the algorithm for constructing the Chinese postman problem works, apply the algorithm and justify their choice of algorithm.
Travelling salesman problem to determine the Hamiltonian cycle of least weight in a weighted complete graph. Nearest neighbour algorithm for determining an upper bound for the travelling salesman problem. Deleted vertex algorithm for determining a lower bound for the travelling salesman problem.	Practical problems should be converted to the classical problem by completion of a table of least distances where necessary.

Connections

Other contexts: Using GPS to find the shortest route home; describe current and voltage in circuits as cycles; vehicle routing problems.

International-mindedness: The "Bridges of Konigsberg" problem; the Chinese postman problem was first posed by the Chinese mathematician Kwan Mei-Ko in 1962.

TOK: What practical problems can or does mathematics try to solve? Why are problems such as the travelling salesman problem so enduring? What does it mean to say the travelling salesman problem is "NP hard"?

Download connections template

Topic 4: Statistics and probability

Concepts

Essential understandings:

Statistics is concerned with the collection, analysis and interpretation of quantitative data and uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned, in detail, to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk and to make predictions about seemingly random events.

Suggested concepts embedded in this topic:

Quantity, validity, approximation, modelling, relationships, patterns.

AHL: Systems, representation.

Content-specific conceptual understandings:

- Organizing, representing, analysing and interpreting data, and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Correlation and regression are powerful tools for identifying patterns and equivalence of systems.
- Modelling and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a representation of the relationship between the theory and reality, allowing us to make predictions about what might happen.

AHL

- Statistical literacy involves identifying reliability and validity of samples and whole populations in a closed system.
- A systematic approach to hypothesis testing allows statistical inferences to be tested for validity.
- Representation of probabilities using transition matrices enables us to efficiently predict long-term behaviour and outcomes.

SL content

Recommended teaching hours: 36

The aim of the standard level content in the statistics and probability topic is to introduce students to important concepts, techniques and representations used in statistics and probability and their meaningful application in the real world. Students should be given the opportunity to approach this topic in a practical way, to understand why certain techniques are used and to interpret the results. The use of technology such as simulations, spreadsheets, statistics software and statistics apps can greatly enhance this topic.

It is expected that most of the calculations required will be carried out using technology, but explanations of calculations by hand may enhance understanding. The emphasis is on choosing the most appropriate technique, and understanding and interpreting the results obtained in context.

In examinations students should be familiar with how to use the statistics functionality of allowed technology.

At SL the data set will be considered to be the population unless otherwise stated.

Sections SL4.1 to SL4.9 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 4.1

Content	**Guidance, clarification and syllabus links**
Concepts of population, sample, random sample, discrete and continuous data.	This is designed to cover the key questions that students should ask when they see a data set/analysis.
Reliability of data sources and bias in sampling.	Dealing with missing data, errors in the recording of data.
Interpretation of outliers.	Outlier is defined as a data item which is more than $1.5 \times$ interquartile range (IQR) from the nearest quartile. Awareness that, in context, some outliers are a valid part of the sample but some outlying data items may be an error in the sample. **Link to**: box and whisker diagrams (SL4.2) and measures of dispersion (SL4.3).
Sampling techniques and their effectiveness.	Simple random, convenience, systematic, quota and stratified sampling methods.

Connections

Links to other subjects: Descriptive statistics and random samples (biology, psychology, sports exercise and health science, environmental systems and societies, geography, economics; business management); research methodologies (psychology).

Aim 8: Misleading statistics; examples of problems caused by absence of representative samples, for example Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston "pot-hole" app.

International-mindedness: The Kinsey report–famous sampling techniques.

TOK: Why have mathematics and statistics sometimes been treated as separate subjects? How easy is it to be misled by statistics? Is it ever justifiable to purposely use statistics to mislead others?

Download connections template

SL 4.2

Content	**Guidance, clarification and syllabus links**
Presentation of data (discrete and continuous): frequency distributions (tables).	Class intervals will be given as inequalities, without gaps.
Histograms. Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles, range and interquartile range (IQR).	Frequency histograms with equal class intervals. **Not required:** Frequency density histograms.
Production and understanding of box and whisker diagrams.	Use of box and whisker diagrams to compare two distributions, using symmetry, median, interquartile range or range. Outliers should be indicated with a cross. Determining whether the data may be normally distributed by consideration of the symmetry of the box and whiskers.

Connections

Links to other subjects: Presentation of data (sciences, individuals and societies).

International-mindedness: Discussion of the different formulae for the same statistical measure (for example, variance).

TOK: What is the difference between information and data? Does "data" mean the same thing in different areas of knowledge?

Download connections template

SL 4.3

Content	Guidance, clarification and syllabus links
Measures of central tendency (mean, median and mode). Estimation of mean from grouped data.	Calculation of mean using formula and technology. Students should use mid-interval values to estimate the mean of grouped data.
Modal class.	For equal class intervals only.
Measures of dispersion (interquartile range, standard deviation and variance).	Calculation of standard deviation and variance of the sample using only technology, however hand calculations may enhance understanding. Variance is the square of the standard deviation.
Effect of constant changes on the original data.	**Examples:** If three is subtracted from the data items, then the mean is decreased by three, but the standard deviation is unchanged. If all the data items are doubled, the mean is doubled and the standard deviation is also doubled.
Quartiles of discrete data.	Using technology. Awareness that different methods for finding quartiles exist and therefore the values obtained using technology and by hand may differ.

Connections

Other contexts: Comparing variation and spread in populations, human or natural, for example agricultural crop data, social indicators, reliability and maintenance.

Links to other subjects: Descriptive statistics (sciences and individuals and societies); consumer price index (economics).

International-mindedness: The benefits of sharing and analysing data from different countries; discussion of the different formulae for variance.

TOK: Could mathematics make alternative, equally true, formulae? What does this tell us about mathematical truths? Does the use of statistics lead to an over-emphasis on attributes that can be easily measured over those that cannot?

Download connections template

SL 4.4

Content	Guidance, clarification and syllabus links
Linear correlation of bivariate data. Pearson's product-moment correlation coefficient, *r*.	Technology should be used to calculate *r*. However, hand calculations of *r* may enhance understanding. Critical values of *r* will be given where appropriate. Students should be aware that Pearson's product moment correlation coefficient (*r*) is only meaningful for linear relationships.
Scatter diagrams; lines of best fit, by eye, passing through the mean point.	Positive, zero, negative; strong, weak, no correlation. Students should be able to make the distinction between correlation and causation and know that correlation does not imply causation.
Equation of the regression line of y on x. Use of the equation of the regression line for prediction purposes. Interpret the meaning of the parameters, a and b, in a linear regression $y=ax+b$.	Technology should be used to find the equation. Students should be aware: • of the dangers of extrapolation • that they cannot always reliably make a prediction of x from a value of y, when using a y on x line.

Connections

Other contexts: Linear regressions where correlation exists between two variables. Exploring cause and dependence for categorical variables, for example, on what factors might political persuasion depend?

Links to other subjects: Curves of best fit, correlation and causation (sciences); scatter graphs (geography).

Aim 8: The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.

TOK: Correlation and causation–can we have knowledge of cause and effect relationships given that we can only observe correlation? What factors affect the reliability and validity of mathematical models in describing real-life phenomena?

Download connections template

SL 4.5

Content	Guidance, clarification and syllabus links
Concepts of trial, outcome, equally likely outcomes, relative frequency, sample space (U) and event. The probability of an event A is $\mathrm{P}(A)=\frac{n(A)}{n(U)}$. The complementary events A and A' (not A).	Sample spaces can be represented in many ways, for example as a table or a list. Experiments using coins, dice, cards and so on, can enhance understanding of the distinction between experimental (relative frequency) and theoretical probability. Simulations may be used to enhance this topic.
Expected number of occurrences.	**Example:** If there are 128 students in a class and the probability of being absent is 0.1, the expected number of absent students is 12.8.

Connections

Other contexts: Actuarial studies and the link between probability of life spans and insurance premiums, government planning based on likely projected figures, Monte Carlo methods.

Links to other subjects: Theoretical genetics and Punnett squares (biology); the position of a particle (physics).

Aim 8: The ethics of gambling.

International-mindedness: The St Petersburg paradox; Chebyshev and Pavlovsky (Russian).

TOK: To what extent are theoretical and experimental probabilities linked? What is the role of emotion in our perception of risk, for example in business, medicine and travel safety?

Use of technology: Computer simulations may be useful to enhance this topic.

Download connections template

SL 4.6

Content	**Guidance, clarification and syllabus links**
Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities.	
Combined events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Mutually exclusive events: $\mathrm{P}(A \cap B) = 0$.	The non-exclusivity of “or”.
Conditional probability: $\mathrm{P}(A\|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.	An alternate form of this is: $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A\|B)$. Problems can be solved with the aid of a Venn diagram, tree diagram, sample space diagram or table of outcomes without explicit use of formulae. Probabilities with and without replacement.
Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.	

Connections

Aim 8: The gambling issue: use of probability in casinos. Could or should mathematics help increase incomes in gambling?

TOK: Can calculation of gambling probabilities be considered an ethical application of mathematics? Should mathematicians be held responsible for unethical applications of their work?

Download connections template

SL 4.7

Content	Guidance, clarification and syllabus links
Concept of discrete random variables and their probability distributions. Expected value (mean), $\mathrm{E}(X)$ for discrete data. Applications.	Probability distributions will be given in the following ways: $\begin{array}{cccccc} X & 1 & 2 & 3 & 4 & 5 \\ P(X=x) & 0.1 & 0.2 & 0.15 & 0.05 & 0.5 \end{array}$ $\mathrm{P}(X=x)=\frac{1}{18}(4+x)$ for $x \in \{1, 2, 3\}$; $\mathrm{E}(X)=0$ indicates a fair game where X represents the gain of a player.

Connections

Other contexts: Games of chance.

Aim 8: Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (for example, economics)?

TOK: What do we mean by a “fair” game? Is it fair that casinos should make a profit?

Download connections template

SL 4.8

Content	Guidance, clarification and syllabus links
Binomial distribution. Mean and variance of the binomial distribution.	Situations where the binomial distribution is an appropriate model. In examinations, binomial probabilities should be found using available technology. **Not required:** Formal proof of mean and variance. **Link to:** expected number of occurrences (SL4.5).

Connections

Aim 8: Pascal's triangle, attributing the origin of a mathematical discovery to the wrong mathematician.

International-mindedness: The so-called "Pascal's triangle" was known to the Chinese mathematician Yang Hui much earlier than Pascal.

TOK: What criteria can we use to decide between different models?

Enrichment: Hypothesis testing using the binomial distribution.

Download connections template

SL 4.9

Content	Guidance, clarification and syllabus links
The normal distribution and curve. Properties of the normal distribution. Diagrammatic representation.	Awareness of the natural occurrence of the normal distribution. Students should be aware that approximately 68% of the data lies between $\mu \pm \sigma$, 95% lies between $\mu \pm 2\sigma$ and 99.7% of the data lies between $\mu \pm 3\sigma$.
Normal probability calculations.	Probabilities and values of the variable must be found using technology.
Inverse normal calculations	For inverse normal calculations mean and standard deviation will be given. This does not involve transformation to the standardized normal variable z.

Connections

Links to other subjects: Normally distributed real-life measurements and descriptive statistics (sciences, psychology, environmental systems and societies)

Aim 8: Why might the misuse of the normal distribution lead to dangerous inferences and conclusions?

International-mindedness: De Moivre’s derivation of the normal distribution and Quetelet’s use of it to describe *l’homme moyen*.

TOK: To what extent can we trust mathematical models such as the normal distribution? How can we know what to include, and what to exclude, in a model?

Download connections template

SL 4.10

Content	Guidance, clarification and syllabus links
Spearman's rank correlation coefficient, r_s.	In examinations Spearman's rank correlation coefficient, r_s, should be found using technology. If data items are equal, ranks should be averaged.
Awareness of the appropriateness and limitations of Pearson's product moment correlation coefficient and Spearman's rank correlation coefficient, and the effect of outliers on each.	Students should be aware that Pearson's product moment correlation coefficient is useful when testing for only linearity and Spearman's correlation coefficient for any monotonic relationship. Spearman's correlation coefficient is less sensitive to outliers than Pearson's product moment correlation coefficient. **Not required:** Derivation/proof of Pearson's product moment correlation coefficient and Spearman's rank correlation coefficient.

Connections

Links to other subjects: Fieldwork (biology, psychology, environmental systems and societies, sports exercise and health science)

Aim 8: The physicist Frank Oppenheimer wrote: "Prediction is dependent only on the assumption that observed patterns will be repeated". This is the danger of extrapolation. There are many examples of its failure in the past, for example share prices, the spread of disease, climate change.

TOK: Does correlation imply causation? Mathematics and the world. Given that a set of data may be approximately fitted by a range of curves, where would a mathematician seek for knowledge of which equation is the "true" model?

Links to websites: www.wikihow.com/Calculate-Spearman%27s-Rank-Correlation-Coefficient

External website: Use of databases such as Gapminder.

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SL 4.11

Content	Guidance, clarification and syllabus links
Formulation of null and alternative hypotheses, H_0 and H_1. Significance levels. p -values.	Students should express H_0 and H_1 as an equation or inequality, or in words as appropriate.
Expected and observed frequencies. The χ^2 test for independence: contingency tables, degrees of freedom, critical value. The χ^2 goodness of fit test.	In examinations: • the maximum number of rows or columns in a contingency table will be 4 • the degrees of freedom will always be greater than one. At SL the degrees of freedom for the goodness of fit test will always be $n-1$ • the χ^2 critical value will be given if appropriate • students will be expected to use technology to find a p -value and the χ^2 statistic • only questions on upper tail tests with commonly-used significance levels (1%, 5%, 10%) will be set • students will be expected to either compare a p -value to the given significance level or compare the χ^2 statistic to a given critical value • expected frequencies will be greater than 5. Hand calculations of the expected values or the χ^2 statistic may enhance understanding.

	If using χ^2 tests in the IA, students should be aware of the limitations of the test for expected frequencies of 5 or less.
The t-test. Use of the p-value to compare the means of two populations. Using one-tailed and two-tailed tests.	In examinations calculations will be made using technology. At SL, samples will be unpaired, and population variance will always be unknown. Students will be asked to interpret the results of a test. Students should know that the underlying distribution of the variables must be normal for the t-test to be applied. In examinations, students should assume that variance of the two groups is equal and therefore the pooled two-sample t-test should be used.

Connections

Other contexts: Psychology: A common test is the Mann-Whitney U test. When and why is this thought to be a more reliable test in psychology?

Links to other subjects: Fieldwork (biology, psychology, environmental systems and societies, sports exercise and health science, geography).

TOK: Why have some research journals "banned" p-values from their articles because they deem them too misleading? In practical terms, is saying that a result is significant the same as saying it is true? How is the term "significant" used differently in different areas of knowledge?

Use of technology: Use of simulations to generate data.

Enrichment: When performing a χ^2 test Yates continuity correction is often applied to small samples. Is it universally accepted as a valid method? In what situations would you use Yates and why? Are there other ways to deal with small sample sizes?

Download connections template

AHL content

Recommended teaching hours: 16

The aim of the AHL content in the statistics and probability topic is to extend and build upon the aims, concepts and skills from the standard level content. It allows students to develop skills in the design of data collection methods taking consideration of validity and reliability, regression is extended to non-linear situations, concepts involving samples and populations are introduced and students will develop their skills in deciding which tests to use in context. Students will be introduced to transition matrices and establish the links between matrices, probability and eigenvalues.

It is expected that students will be able to choose appropriate techniques and interpret their results. Students are expected to set up a problem mathematically and then calculate the answers using technology. Technology-specific language should not be used within these explanations.

AHL 4.12

Content	Guidance, clarification and syllabus links
Design of valid data collection methods, such as surveys and questionnaires. Selecting relevant variables from many variables. Choosing relevant and appropriate data to analyse.	Biased and unbiased, personal, unstructured and structured (with consistent answer choices), and precise questioning.
Categorizing numerical data in a χ^2 table and justifying the choice of categorisation. Choosing an appropriate number of degrees of freedom when estimating parameters from data when carrying out the χ^2 goodness of fit test.	Appropriate categories should be chosen with expected frequencies greater than 5.
Definition of reliability and validity. Reliability tests. Validity tests.	Students should understand the difference between reliability and validity and be familiar with the following methods: Reliability: Test-retest, parallel forms. Validity: Content, criterion-related.

Connections

Other contexts: Data from social media behaviour and algorithms.

Links to other subjects: Data collection in field work (biology, psychology, environmental systems and societies, sports exercise and health science, geography, business management and design technology); data from social media and marketing sources (business management)

TOK: What are the strengths and limitations of different methods of data collection, such as questionnaires?

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AHL 4.13

Content	Guidance, clarification and syllabus links
Non-linear regression.	**Link to:** geometric sequences and series (SL1.3).
Evaluation of least squares regression curves using technology.	In examinations, questions may be asked on linear, quadratic, cubic, exponential, power and sine regression.
Sum of square residuals (SS_{res}) as a measure of fit for a model.	
The coefficient of determination (R^2). Evaluation of R^2 using technology.	R^2 gives the proportion of variability in the second variable accounted for by the chosen model. Awareness that $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$ and hence = 1 if $SS_{res} = 0$, may enhance understanding but will not be examined. Awareness that many factors affect the validity of a model and the coefficient of determination, by itself, is not a good way to decide between different models. The connection between the coefficient of determination and the Pearson's product moment correlation coefficient for linear models.

Connections

Links to other subjects: Evaluation of R^2 in graphical analysis (sciences).

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AHL 4.14

Content	Guidance, clarification and syllabus links
Linear transformation of a single random variable.	$\mathrm{Var}(X)$ is the expected variance of the random variable X. Variance formula will not be required in examinations. $\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$. $\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$.
Expected value of linear combinations of *n* random variables. Variance of linear combinations of *n* independent random variables.	
$\overline{x}$ as an unbiased estimate of μ.	$\overline{x}=\sum_{i=1}^{n}\frac{x_i}{n}$.
s_{n-1}^2 as an unbiased estimate of σ^2.	$s_{n-1}^2=\frac{n}{n-1}s_n^2=\sum_{i=1}^{k}\frac{f_i(x_i-\overline{x})^2}{n-1}$, where $n=\sum_{i=1}^{k}f_i$. Demonstration that $\mathrm{E}(\overline{X})=\mu$ and $\mathrm{E}(s_{n-1}^2)=\sigma^2$ will not be examined, but may help understanding.

Connections

TOK: Mathematics and the world: In the absence of knowing the value of a parameter, will an unbiased estimator always be better than a biased one?

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AHL 4.15

Content	Guidance, clarification and syllabus links
A linear combination of n independent normal random variables is normally distributed. In particular, $X \sim \mathrm{N}(\mu, \quad \sigma^2) \Rightarrow \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.	
Central limit theorem.	In general, $\overline{X}$ approaches normality for large n, how large depends upon the distribution from which the sample is taken. In examinations, $n > 30$ will be considered sufficient. Online simulations are useful for visualisation.

Connections

Links to other subjects: Data from multiple samples in field studies (sciences, and individuals and societies).

Aim 8: Mathematics and the world. “Without the central limit theorem, there could be no statistics of any value within the human sciences”.

TOK: The central limit theorem can be proved mathematically (formalism), but its truth can be confirmed by its applications (empiricism). What does this suggest about the nature and methods of mathematics?

Enrichment: For a normally distributed population of size N, how many random samples of size n do you need to take in order to verify the central limit theorem?

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AHL 4.16

Content	Guidance, clarification and syllabus links
Confidence intervals for the mean of a normal population.	Students should be able to interpret the meaning of their results in context. Use of the normal distribution when σ is known and the t-distribution when σ is unknown, regardless of sample size.

Connections

Other contexts: Forecasting–attaching value to claims and predictions.

Links to other subjects: Analysis of data from field studies (sciences and individuals and societies).

TOK: Mathematics and the world. Claiming brand A is "better" on average than brand B can mean very little if there is a large overlap between the confidence intervals of the two means.

Download connections template

AHL 4.17

Content	Guidance, clarification and syllabus links
Poisson distribution, its mean and variance. Sum of two independent Poisson distributions has a Poisson distribution.	Situations in which it is appropriate to use a Poisson distribution as a model: 1. Events are independent 2. Events occur at a uniform average rate (during the period of interest). Given a context, students should be able to select between the normal, the binomial and the Poisson distributions, recognizing where a particular distribution is appropriate. **Not required:** Formal proof of means and variances for probability distributions.

Connections

Other contexts: Telecommunications, call management, traffic management, biological mutations, emergency room admissions, typos in publications.

TOK: To what extent can mathematical models such as the Poisson distribution be trusted? What role do mathematical models play in other areas of knowledge?

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AHL 4.18

Content	**Guidance, clarification and syllabus links**
Critical values and critical regions. Test for population mean for normal distribution.	Use of the normal distribution when σ is known and the t-distribution when σ is unknown, regardless of sample size. Samples may be paired or unpaired. The case of matched pairs is to be treated as an example of a single sample technique. Students will not be expected to calculate critical regions for t-tests.
Test for proportion using binomial distribution.	
Test for population mean using Poisson distribution.	Poisson and binomial tests will be one-tailed only.
Use of technology to test the hypothesis that the population product moment correlation coefficient (ρ) is 0 for bivariate normal distributions.	In examinations the data will be given.
Type I and II errors including calculations of their probabilities.	Applied to normal with known variance, Poisson and binomial distributions. For discrete random variables, hypothesis tests and critical regions will only be required for one-tailed tests. The critical region will maximize the probability of a Type I error while keeping it less than the stated significance level.

Connections

Links to other subjects: Field studies (sciences and individuals and societies).

TOK: Mathematics and the world. In practical terms, is saying that a result is significant the same as saying that it is true? Mathematics and the world. Does the ability to test only certain parameters in a population affect the way knowledge claims in the human sciences are valued? When is it more important not to make a Type I error and when is it more important not to make a Type II error?

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AHL 4.19

Content	**Guidance, clarification and syllabus links**
Transition matrices. Powers of transition matrices.	In general, the column state matrix ($\boldsymbol{s}_n$) after n transitions is given by $\boldsymbol{s}_n = \boldsymbol{T}^n \boldsymbol{s}_0$, where $\boldsymbol{T}$ is the transition matrix, with $\boldsymbol{T}_{ij}$ representing the probability of moving from state j to state i, and $\boldsymbol{s}_0$ is the initial state matrix. Use of transition diagrams to represent transitions in discrete dynamical systems.
Regular Markov chains. Initial state probability matrices.	
Calculation of steady state and long-term probabilities by repeated multiplication of the transition matrix or by solving a system of linear equations.	Examination questions will state when exact solutions obtained from solving equations are required. Awareness that the solution is the eigenvector corresponding to the eigenvalue equal to 1. **Link to:** matrices (AHL1.14), eigenvalues (AHL1.15) and adjacency matrices (AHL3.15).

Connections

Other contexts: Absorbing states for Markov chains, the gambler's ruin problem.

Website: Simulation for Markov chains: setosa.io/blog/2014/07/26/markov-chains/

Enrichment: Leslie matrices are used extensively in biology.

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Topic 5: Calculus

Concepts

Essential understandings:

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behaviour of functions and allows us to interpret the features of their graphs.

Suggested concepts embedded in this topic:

Change, patterns, relationships, approximation, space, generalization.

AHL: Systems, quantity.

Content-specific conceptual understandings:

- Students will understand the links between the derivative and the rate of change and interpret the meaning of this in context.
- Students will understand the relationship between the integral and area and interpret the meaning of this in context.
- Finding patterns in the derivatives of polynomials and their behavior, such as increasing or decreasing, allows a deeper appreciation of the properties of the function at any given point or instant.
- Calculus is a concise form of communication used to approximate nature.
- Numerical integration can be used to approximate areas in the physical world.

- Optimization of a function allows us to find the largest or smallest value that a function can take in general and can be applied to a specific set of conditions to solve problems.
- Maximum and minimum points help to solve optimization problems.
- The area under a function on a graph has a meaning and has applications in space and time.

AHL

- Kinematics allows us to describe the motion and direction of objects in closed systems in terms of displacement, velocity, and acceleration.
- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities.
- Phase portraits enable us to visualize the behavior of dynamic systems.

SL content

Recommended teaching hours: 19

The aim of the standard level content in the calculus topic is to introduce students to the key concepts and techniques of differential and integral calculus and their use to approach practical problems.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of calculus.

Sections SL5.1 to SL5.5 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 5.1

Content	Guidance, clarification and syllabus links
Introduction to the concept of a limit.	Estimation of the value of a limit from a table or graph. **Not required:** Formal analytic methods of calculating limits.
Derivative interpreted as gradient function and as rate of change.	Forms of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $f'(x)$, $\frac{\mathrm{d}V}{\mathrm{d}r}$ or $\frac{\mathrm{d}s}{\mathrm{d}t}$ for the first derivative. Informal understanding of the gradient of a curve as a limit.

Connections

Links to other subjects: Marginal cost, marginal revenue, marginal profit, market structures (economics); kinematics, induced emf and simple harmonic motion (physics); interpreting the gradient of a curve (chemistry)

Aim 8: The debate over whether Newton or Leibnitz discovered certain calculus concepts; how the Greeks' distrust of zero meant that Archimedes' work did not lead to calculus.

International-mindedness: Attempts by Indian mathematicians (500-1000 CE) to explain division by zero.

TOK: What value does the knowledge of limits have? Is infinitesimal behaviour applicable to real life? Is intuition a valid way of knowing in mathematics?

Use of technology: Spreadsheets, dynamic graphing software and GDC should be used to explore ideas of limits, numerically and graphically. Hypotheses can be formed and then tested using technology.

Download connections template

SL 5.2

Content	Guidance, clarification and syllabus links
Increasing and decreasing functions. Graphical interpretation of $f'(x) > 0, f'(x) = 0, f'(x) < 0$.	Identifying intervals on which functions are increasing $(f'(x) > 0)$ or decreasing $(f'(x) < 0)$.

Connections

Download connections template

SL 5.3

Content	Guidance, clarification and syllabus links
Derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}, n \in Z$ The derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$ where all exponents are integers.	

Connections

TOK: The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats such as getting a man on the Moon. What does this tell us about the links between mathematical models and reality?

Download connections template

SL 5.4

Content	Guidance, clarification and syllabus links
Tangents and normals at a given point, and their equations.	Use of both analytic approaches and technology.

Connections

Links to other subjects: Instantaneous velocity and optics, equipotential surfaces (physics); price elasticity (economics).

TOK: In what ways has technology impacted how knowledge is produced and shared in mathematics? Does technology simply allow us to arrange existing knowledge in new and different ways, or should this arrangement itself be considered knowledge?

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SL 5.5

Content	Guidance, clarification and syllabus links
Introduction to integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where $n \in Z$, $n \neq -1$.	Students should be aware of the link between anti-derivatives, definite integrals and area.
Anti-differentiation with a boundary condition to determine the constant term.	**Example:** If $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 1$, then $y = x^3 + \frac{1}{2}x^2 + 8.5$.
Definite integrals using technology. Area of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x) > 0$.	Students are expected to first write a correct expression before calculating the area, for example $\int_2^6 (3x^2 + 4)\mathrm{d}x$. The use of dynamic geometry or graphing software is encouraged in the development of this concept.

Connections

Other contexts: Velocity-time graphs

Links to other subjects: Velocity-time and acceleration-time graphs (physics and sports exercise and health science)

TOK: Is it possible for an area of knowledge to describe the world without transforming it?

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SL 5.6

Content	Guidance, clarification and syllabus links
Values of x where the gradient of a curve is zero. Solution of $f'(x) = 0$. Local maximum and minimum points.	Students should be able to use technology to generate $f'(x)$ given $f(x)$, and find the solutions of $f'(x) = 0$. Awareness that the local maximum/minimum will not necessarily be the greatest/least value of the function in the given domain.

Connections

Other contexts: Profit, area, volume, cost.

Links to other subjects: Displacement-time and velocity-time graphs and simple harmonic motion graphs (physics).

TOK: Is it possible for an area of knowledge to describe the world without transforming it?

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SL 5.7

Content	Guidance, clarification and syllabus links
Optimisation problems in context.	**Examples:** Maximizing profit, minimizing cost, maximizing volume for a given surface area. In SL examinations, questions on kinematics will not be set.

Connections

Other contexts: Efficient use of material in packaging.

Links to other subjects: Kinematics (physics); allocative efficiency (economics).

TOK: How can the rise in tax for plastic containers, for example plastic bags, plastic bottles etc be justified using optimization?

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SL 5.8

Content	Guidance, clarification and syllabus links
Approximating areas using the trapezoidal rule.	Given a table of data or a function, make an estimate for the value of an area using the trapezoidal rule, with intervals of equal width. **Link to:** upper and lower bounds (SL1.6) and areas under curves (SL5.5).

Connections

Other contexts: Irregular areas that are not described by mathematical functions, for example lakes.

Links to other subjects: Kinematics (Physics).

Use of technology: Use dynamic graphing software to calculate the approximate area under a curve and interpret its meaning.

Enrichment: Exploring other numerical integration techniques such as Simpson's rule.

Download connections template

AHL content

Recommended teaching hours: 22 hours

The aim of the AHL content in the calculus topic is to extend and build upon the aims, concepts and skills from the standard level content. Further differential and integral calculus techniques are introduced to enable students to model and interpret practical contexts.

AHL 5.9

Content	Guidance, clarification and syllabus links
The derivatives of sin x, cos x, tan x, e^x, ln x, x^n where $n \in Q$. The chain rule, product rule and quotient rules. Related rates of change.	**Link to**: maximum and minimum points (SL5.6) and optimisation (SL5.7).

Connections

Links to other subjects: Uniform circular motion and induced emf (physics).

TOK: Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work. What does this suggest about the nature of progress and development in mathematics? How might this be similar/different to the nature of progress and development in other areas of knowledge?

Download connections template

AHL 5.10

Content	Guidance, clarification and syllabus links
The second derivative.	Both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$ for the second derivative.
Use of second derivative test to distinguish between a maximum and a minimum point.	Awareness that a point of inflexion is a point at which the concavity changes and interpretation of this in context.
	Use of the terms “concave-up” for $f''(x) > 0$, and “concave-down” for $f''(x) < 0$. **Link to:** kinematics (AHL5.13) and second order differential equations (AHL5.18).

Connections

Links to other subjects: Simple harmonic motion (physics).

TOK: Music can be expressed using mathematics. Does this mean that music is mathematical/that mathematics is musical?

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AHL 5.11

Content	Guidance, clarification and syllabus links
Definite and indefinite integration of x^n where $n \in Q$, including $n = -1$, sin x, cos x, $\frac{1}{\cos^2 x}$ and e^x.	
Integration by inspection, or substitution of the form $\int f(g(x))g'(x)dx$.	**Examples:** $\int \sin(2x+5)dx$, $\int \frac{1}{3x+2}dx$, $\int 4x\sin x^2\, dx$, $\int \frac{\sin x}{\cos x}\, dx$.

Connections

International-mindedness: The successful calculation of the volume of a pyramidal frustrum by ancient Egyptians (the Egyptian Moscow mathematical papyrus).

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AHL 5.12

Content	Guidance, clarification and syllabus links
Area of the region enclosed by a curve and the x or y-axes in a given interval.	Including negative integrals.
Volumes of revolution about the x- axis or y-axis.	$V = \int_a^b \pi y^2 dx$ or $V = \int_a^b \pi x^2 dy$

Connections

Other contexts: Industrial design; architecture.

International-mindedness: Accurate calculation of the volume of a cylinder by Chinese mathematician Liu Hui; use of infinitesimals by Greek geometers; Ibn Al Haytham, the first mathematician to calculate the integral of a function in order to find the volume of a paraboloid.

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AHL 5.13

Content	Guidance, clarification and syllabus links
Kinematic problems involving displacement s, velocity v and acceleration a.	$v = \frac{\mathrm{d}s}{\mathrm{d}t}$; $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. Displacement $= \int_{t_1}^{t_2} v(t)\mathrm{d}t$. Total distance travelled $= \int_{t_1}^{t_2} \lvert v(t)\rvert \mathrm{d}t$. Speed is the magnitude of velocity. Use of $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$.

Connections

Links to other subjects: Kinematics (physics).

International-mindedness: Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage? Who decides what is mathematics?

TOK: What is the role of convention in mathematics? Is this similar or different to the role of convention in other areas of knowledge?

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AHL 5.14

Content	Guidance, clarification and syllabus links
Setting up a model/differential equation from a context.	**Example:** The growth of an algae G, at time t, is proportional to $\sqrt{G}$.
Solving by separation of variables.	**Example:** An exponential model as a solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$. The term “general solution”.

Connections

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AHL 5.15

Content	Guidance, clarification and syllabus links
Slope fields and their diagrams.	Students will be required to use and interpret slope fields.

Connections

TOK: In what ways do values affect our representations of the world, for example in statistics, maps, visual images or diagrams?

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AHL 5.16

Content	Guidance, clarification and syllabus links
Euler’s method for finding the approximate solution to first order differential equations. Numerical solution of $\frac{dy}{dx}=f(x, y)$.	Spreadsheets should be used to find approximate solutions to differential equations. In examinations, values will be generated using permissible technology.
Numerical solution of the coupled system $\frac{dx}{dt}=f_1(x, y, t)$ and $\frac{dy}{dt}=f_2(x, y, t)$.	Contexts could include predator-prey models.

Connections

Other contexts: The SIR model for infection as an extension of the method; Lotka-Volterra models.

TOK: To what extent is certainty attainable in mathematics? Is certainty attainable, or desirable, in other areas of knowledge?

Enrichment: Runge-Kutta methods.

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AHL 5.17

Content	Guidance, clarification and syllabus links
Phase portrait for the solutions of coupled differential equations of the form: $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy.$ Qualitative analysis of future paths for distinct, real, complex and imaginary eigenvalues. Sketching trajectories and using phase portraits to identify key features such as equilibrium points, stable populations and saddle points.	Systems will have distinct, non-zero, eigenvalues. If the eigenvalues are: • Positive or complex with positive real part, all solutions move away from the origin • Negative or complex with negative real part, all solutions move towards the origin • Complex, the solutions form a spiral • Imaginary, the solutions form a circle or ellipse • Real with different signs (one positive, one negative) the origin is a saddle point. Calculation of exact solutions is only required for the case of real distinct eigenvalues. **Link to:** eigenvectors and eigenvalues (AHL1.15).

Connections

Other contexts: Jacobian matrix is used to investigate the stability of equilibrium states for non-linear differential equations.

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AHL 5.18

Content	Guidance, clarification and syllabus links
Solutions of $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$ by Euler's method.	Write as coupled first order equations $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = f(x, y, t)$. Solutions of $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b = 0$, can also be investigated using the phase portrait method in AHL 5.17 above. Understanding the occurrence of simple second order differential equations in physical phenomena would aid understanding but in examinations the equation will be given.

Connections

TOK: How have notable individuals such as Euler shaped the development of mathematics as an area of knowledge?

Use of technology: Use of spreadsheets to generate values.

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